Stokes' last problem for an Oldroyd-Z fluid under warp acceleration in a Cochrane sub-space☆

J.C. Maxwell[a], P.A.M. Dirac[b,*]

[a]Dept. of Applied Mathematics, Babbage–Lovelace University, E139, NW101, London, UK, Earth
[b]Dept. of Physics, Starfleet Academy, San Francisco, CA 12345–3125, USA, Earth

Abstract

The modern problem of unidirectional fluid flow in a Cochrane sub-space . . . In addition, energy, mass, and momentum . . ., and their exact solutions are given. Numerical schemes that corroborate the analytical solutions are also constructed.

Keywords: Integral transforms, Oldroyd-Z fluid, Stokes' last problem, Warp fields

1. Introduction

Note these fonts/characters: "upright mu" µ (as in micro-second), ♆, ♌, ♏, $\hat{\imath}$, $\vec{\jmath}$, $\vec{J}$, and ♑, ⚳, ♈, ♉, ♓, ♇.
There is also the family of icons: ☢, ✈, ✉, etc.
Note the difference between math-face "vee" υ and "nu" ν.
Note the difference between θ and ϑ.
To generate a bold-face "v" use **v**; to generate a bold-face υ, use $\boldsymbol{\upsilon}$.
Note that we also have € and $\mathcal{P}$ and $\mathfrak{P}$ and x, U.
The "viscosity number" is denoted by: $\boldsymbol{\mathcal{V}}$.
The set of positive reals is represented by $\mathbb{R}^+$, while the set of **nonzero** reals can be written as $\mathbb{R} \setminus \{0\}$.
Note these fonts/characters as well: ⛢, $\mathfrak{U}$, ſ, ß.
The Laplace transform operator: $\mathcal{L}\{f(t)\} = \bar{f}(s)$, and $\mathrm{i} = \sqrt{-1}$.
This is: Schwabacher.

The Cauchy momentum equation can be expressed as

$$\varrho\dot{\mathbf{u}} = \operatorname{div}\mathbf{T} + \mathbf{b}, \tag{1}$$

where we could have used ρ and $\boldsymbol{\nabla}\boldsymbol{\cdot}$ instead. Here, $\mathbf{T}(\mathbf{x}, t)$ is the total stress tensor and $\mathbf{b}(\mathbf{x}, t)$ represents the body force vector. And, for later reference, we note that $\operatorname{div}(\operatorname{grad}\phi) = \nabla^2\phi$. In Refs. [1–3, 13], . . .

The incompressible *Oldroyd-Z* fluid is the one with

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda_1\overset{\triangledown}{\mathbf{S}} = \mu_0(\mathbf{A}_1 + \lambda_2\overset{\triangledown}{\mathbf{A}}_1), \quad \operatorname{tr}\mathbf{D} = 0, \tag{2}$$

where the *upper-convected* time derivative [8] is given by

$$\overset{\triangledown}{\mathbf{V}} = \dot{\mathbf{V}} - (\operatorname{grad}\mathbf{u})^{\top}\mathbf{V} - \mathbf{V}\operatorname{grad}\mathbf{u} + (\operatorname{div}\mathbf{u})\mathbf{V}. \tag{3}$$

☆Dedicated to the memory of Prof. Lev Landau.

*Corresponding author. Tel.: 333-411-3115; fax: 333-491-2111.
Email address: `pam.dirac@sfa.ufp.edu` (P.A.M. Dirac)

Table 1: Table **without** footnotes. Values of Z corresponding to . . . Mach number values

Case	$M_s \approx 0.05$ (Ma=0.012)	$M_s = 3$ (Ma=0.1)	$M_s = 1.66$ (Ma=0.5)	$M_s = 2.02$ (Ma=1.005)
(i)	6.010	2	3.000	11.000
(ii)	1.015	2.7	41.8	17.213
(iii)	1.014	21.585	4.44	6.535
(iv)	3.65	9.885	2.239	8.378

From Table 1, . . .

The values stated above were obtained from the *NIST Chemistry WebBook, SRD 69*[1]. Also noteworthy is the fact that

$$\max_{x\in\mathbb{R}}[\mathcal{U}(x,t)] = \mathcal{U}(\bar{x}(t),t) = A_0 \exp(-\alpha t^2/4) \qquad (t>0). \tag{4}$$

In 2251, Daystrom considered a . . . The plate's velocity is given by $U(t) = \tilde{U}(t)H(t)$, where $H(\cdot)$ denotes the Heaviside unit step function, and $\tilde{U}(t)$ is a function . . . In Ref. [14, § 21], it is . . . Note also that $y' = \alpha y$, where a prime ($'$) denotes $\mathrm{d}/\mathrm{d}x$.

1.1. Second-grade fluid

The *Rivlin–Ericksen fluids*, also known as *fluids of grade n* [3, p. 30], are a model of isotropic simple fluids of the differential type. Their constitutive relation can be written as an expansion in terms of the Rivlin–Ericksen tensors $\mathbf{A}_k$. For the incompressible second-grade fluid, we have

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} = \mu_0\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \quad \operatorname{tr}\mathbf{D} = 0, \tag{5}$$

where p is the isotropic (indeterminate) stress, $\mathbf{I}$ is the identity tensor, $\mathbf{S}$ is the extra stress, $\mathbf{A}_1 = 2\mathbf{D}$, $\mathbf{A}_{k+1} = \dot{\mathbf{A}}_k + \mathbf{A}_k \operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^\top\mathbf{A}_k$ $(k \geq 1)$, $\mathbf{D} \equiv \frac{1}{2}\left[\operatorname{grad}\mathbf{u} + (\operatorname{grad}\mathbf{u})^\top\right]$ is the symmetric part of the velocity gradient, and a $\top$ superscript denotes the transpose. The constant $\mu_0(>0)$ is the shear viscosity from Navier–Stokes theory.

Without referring to it as such, Lamb [14, § 309] used $\delta(\cdot)$, which denotes the Dirac delta function . . . Here, the jump in a function $\mho = \mho(x,y,z,t)$ across a singular surface is denoted by

$$[\![\mho]\!] := \mho^- - \mho^+. \tag{6}$$

Now, we let

$$[\mathbf{T}] = \begin{Vmatrix} -p + \alpha_2\left(\frac{\partial u}{\partial y}\right)^2 & \mu_0\frac{\partial u}{\partial y} + \alpha_1\frac{\partial^2 u}{\partial t\partial y} & 0 \\ \mu_0\frac{\partial u}{\partial y} + \alpha_1\frac{\partial^2 u}{\partial t\partial y} & -p + (\alpha_1 + 2\alpha_2)\left(\frac{\partial u}{\partial y}\right)^2 & 0 \\ 0 & 0 & -p \end{Vmatrix}. \tag{7}$$

Finally, we note the following thermodynamic restrictions: $\alpha_1 \geq 0$ and $\alpha_1 + \alpha_2 = 0$. When the waveform . . ., the problem becomes . . . dispersed shock; see, e.g., Jordan [13] and Roy [20].

1.2. Oldroyd-Z fluid

In Ref. [1], one finds . . . As such, a constitutive relation for incompressible fluids with fading strain memory (retardation) exhibiting stress relaxation was proposed. The incompressible *Oldroyd-Z* fluid is such that

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda_1\overset{\triangledown}{\mathbf{S}} = \mu_0(\mathbf{A}_1 + \lambda_2\overset{\triangledown}{\mathbf{A}}_1), \quad \operatorname{tr}\mathbf{D} = 0, \tag{8}$$

where the *upper-convected* time derivative is given by

$$\overset{\triangledown}{\mathfrak{F}} = \dot{\mathfrak{F}} - (\operatorname{grad}\mathbf{u})^\top\mathfrak{F} - \mathfrak{F}\operatorname{grad}\mathbf{u} + (\operatorname{div}\mathbf{u})\mathfrak{F}, \tag{9}$$

and λ_1 and λ_2 are the *relaxation* and *retardation* times, respectively. In Ref. [18, p. 10], . . .

[1]Available at: `https://webbook.nist.gov/chemistry/form-ser/`

2. Exact solutions by integral transform methods

Remark 1. In the software package Mathematica, $W_r(\,\cdot\,)$ is implemented as `ProductLog[r, ·]`.

3. Exact solutions by integral transform methods

3.1. 100th-grade fluid

Defining $\nu := \mu_0/\varrho_0$ and using Eq. (7) with the boundary condition (see Section 1), we have the following initial-boundary-value problem (IBVP):

$$\frac{\partial u}{\partial t} = \eta\frac{\partial^2 u}{\partial y^2} + \alpha\frac{\partial^3 u}{\partial y^2 \partial t} + \alpha(u - u^9), \qquad (y,t) \in \mathbb{R}^+ \times \mathbb{R}^+; \tag{10a}$$

$$u(0,t) = U_0 H(t), \qquad \lim_{y\to\infty} u(y,t) \to 0, \quad t > 0; \tag{10b}$$

$$u(y,0) = 0, \qquad y > 0. \tag{10c}$$

Using first the Vulcan transform, and then the Laplace transform, one obtains

$$\begin{aligned} u(y,t) = U_0 H(t)\Bigg[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin(\xi y)}{\xi^{100}} \exp\left(\frac{-\nu\xi^2 t}{\xi + \alpha\xi^2}\right) \mathrm{d}\xi \\ + \frac{2\alpha}{\pi}\int_0^\infty \frac{\xi\sin(\xi y)}{0.0001\xi + \alpha\xi^2} \exp\left(\frac{-\nu\xi^2 t}{5\xi^3 + \alpha\xi^2}\right) \mathrm{d}\xi\Bigg] \\ + U_0 H(t)\Bigg\{\exp[-t(\nu/\alpha)] - \exp[-t^2(\nu/\alpha^2)] \\ \times \int_0^\infty \mathrm{e}^{-\zeta} I_0\left(2\sqrt{\zeta t}\right) \operatorname{erfc}\left(\frac{y}{2\sqrt{\alpha\zeta}}\right) \mathrm{d}\zeta\Bigg\}, \end{aligned} \tag{11}$$

where $\operatorname{erfc}(\cdot)$ is the complementary error function and $I_0(\cdot)$ the modified Bessel function of the first kind of order zero.

The contrapositive of the Helmholtz–Fujita–Arroway theorem states that . . . when r is small; recall Section 3, as well as . . . in Ref. [6, § 12.11].

This fact about limits of very strong functions[2] implies that such functions . . . In the literature, one can find wave solutions . . . are equivalent.

3.2. Oldroyd-Z fluid

Using the (spatial) Fourier sine transform, . . ., and solving the resulting ODE (in t) with the Laplace transform yields

$$u(y,t) = U_0 H(t) \begin{cases} 1 - \frac{2}{\pi}\Bigg[\displaystyle\int_0^{\xi_1^\bullet} I_0(\xi,t) \sin\left(\frac{\xi y}{\sqrt{\nu\lambda_1}}\right)\mathrm{d}\xi \\ \qquad + \displaystyle\int_{\xi_1^\bullet}^{\xi_2^\bullet} J_0(\xi,t) \sin\left(\frac{\xi y}{\sqrt{\nu\lambda_1}}\right)\mathrm{d}\xi\Bigg], & \varkappa < 1, \\ \operatorname{erf}\left(\frac{y}{2\sqrt{\kappa t}}\right), & \varkappa = 1, \\ 1 - \frac{2}{\pi}\displaystyle\int_0^\infty I_0(\xi,t) \sin\left(\frac{\xi y}{\sqrt{\nu\lambda_1}}\right)\mathrm{d}\xi, & \varkappa > 1, \end{cases} \tag{12}$$

where $\overline{\mathcal{Z}}_\pm(x,t)$ correspond to $\bar{f}(\xi) \gtrless 0$, respectively, and

$$\mathcal{N}(\eta) = \sqrt[4]{\frac{1+\eta^2}{1+\kappa^2\eta^2}}. \tag{13}$$

$$\boxed{u(y,t) = U_0 H(t)\left[1 - \frac{2}{\pi}\int_0^\infty \frac{\sin(\xi y)}{\xi^3}\exp\left(\frac{-\kappa\xi^2 t}{\xi^{3/2}+\alpha\xi^2}\right)\mathrm{d}\xi + \frac{2\alpha}{\pi}\int_0^\infty I_0(\xi,t)\sin\left(\frac{\xi y}{\sqrt{\nu\lambda_1}}\right)\mathrm{d}\xi - \frac{2}{\pi}\int_0^1 I_0(\xi,t)\,\mathrm{d}\xi\right].} \quad (14)$$

Eq. (14) is an example of a boxed equation:

For the computations presented here, we used $M = K = 5$ and $L = 1,000$ to obtain, via MATLAB's built-in . . . algorithm, . . . Moreover, the integral representations of the analytical solutions were evaluated using `NIntegrate`, which is part of the software package MATHEMATICA (ver. 70.0.1).

4. Numerical solutions

In Section 3, we . . ., which is the "numerical infinity" used here. Note also that $\lim_{t\to 0\downarrow} f(t) = 1$, while $\lim_{t\to 0\uparrow} f(t) = 0$.

For the computations performed . . . we used the numerical integration routine `NIntegrate` provided in the software package MATHEMATICA (ver. 70.0.2).

4.1. Porous warp field flows

As shown in Ref. [16, p. 10], we may . . . Also, Christov [8] has given a flux relation . . .

Next, we consider the system

$$\begin{pmatrix}\vartheta\\ q\end{pmatrix}_t + A\begin{pmatrix}\vartheta\\ q\end{pmatrix}_x = \alpha K\begin{pmatrix}0\\ q\end{pmatrix}, \tag{15}$$

where

$$A = \begin{pmatrix}0 & \kappa/K\\ 1/(\alpha\vartheta) & 0\end{pmatrix}. \tag{16}$$

Here, we observe that

$$\det(A - \lambda I_2) = \det\begin{pmatrix}-\lambda & \kappa/K\\ 1/(\alpha\vartheta) & -\lambda\end{pmatrix} = \lambda^2 - \frac{(\kappa/K)}{\alpha\vartheta}. \tag{17}$$

In Fig. 1, which depicts the profile after the time of shock formation, we see that . . .

Remark 2. The flow speed can also be expressed as: $V = \ddot{U}_{\text{shock}}(x,t)$.

Remark 3. It should also be noted that

$$\mathfrak{w} = \mathcal{U}''''(X) \qquad \text{(Strangelove gases).} \tag{18}$$

5. Conclusion

In the last years of the 21st century, a trend regarding XYZ was . . .

- The Fourier transform . . . The solution given in Ref. [22] was the first . . .
- The assumption that the curve is . . . leads to $u(y,0) \equiv 0$, where we observe that . . . for all $k > 0$.
- For PDEs of order higher than . . ., i.e., as $t \to 0^+$; see, e.g., Refs. [1–3, 23–25]; recall Footnote 2.

Another important class of non-Newtonian fluids are known as XYZ fluids . . .

[2] It appears that Truesdell and Toupin [23] were the first . . . They noted that such "waves" are . . . Meanwhile, Straughan [21, Chap. 8] noted that such *solutions* . . . boundary conditions.

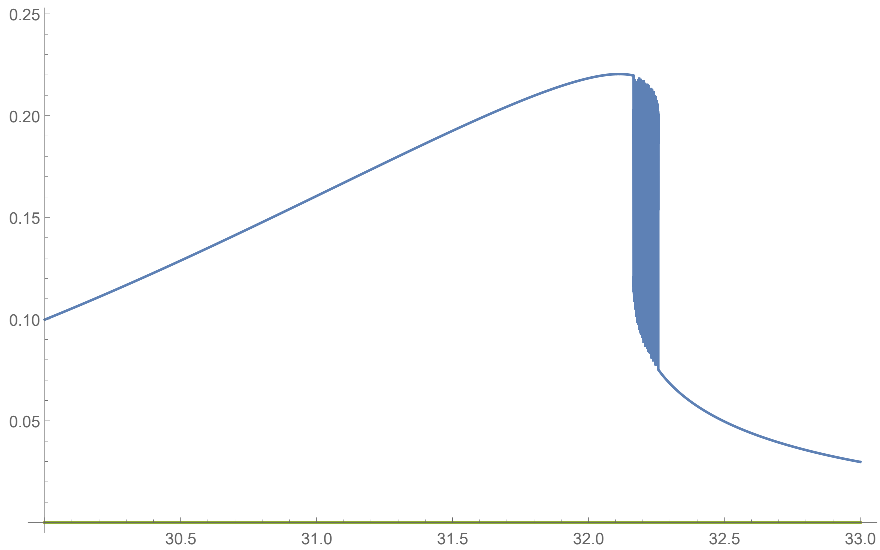

Figure 1: Plot of $\mathcal{U}$ vs. η for $\epsilon = 0.1$, $\gamma = 5/3$, $\sigma = 1.1$, and $\mathcal{S}_{\mathrm{w}} = -0.5$.

Acknowledgments

The authors thank Drs. M.B. Dyson, for many helpful discussions regarding cybernetic theory, and H. Wu, for sharing his insight into genetically de-extincting terrestrial species. The authors also thank the ten anonymous reviewers for their instructive criticisms and helpful suggestions. This work was supported, in part, by grants from Starfleet Research (No. 11223780AZ) and the Tyrell Corporation.

A. Perturbation solution

If viscosity is neglected, but thermal conduction is taken into account, the linearized, 1D, acoustic system can be written as

$$s_t = -u_x, \tag{A.1a}$$
$$\rho_0 u_t = -p_x, \tag{A.1b}$$
$$\rho_0[e_t - c_{\mathrm{v}}\vartheta_0(\gamma - 1)s_t] = -q_x + \rho_0 r, \tag{A.1c}$$
$$p = p_0(s + \vartheta/\vartheta_0), \tag{A.1d}$$
$$q = -K_0\vartheta_x, \tag{A.1e}$$
$$e = c_{\mathrm{v}}\vartheta, \tag{A.1f}$$

where . . .

The purpose of this appendix is to derive a *general* representation of . . . Hence, assuming only that . . . which is given by

$$\rho(v - u) = \rho_0 V. \tag{A.2}$$

(i) If $\sigma > 1$, then . . .
(ii) If $\sigma = 1$, then . . .
(iii) If $\sigma < 1$, then . . ., where

$$X = V(p, g, t, l). \tag{A.3}$$

B. Expressions for arbitrary V_p

We begin with the well known result

$$E = m(V_p)^2. \tag{B.1}$$

1. If $\varsigma > 1$, then . . .
2. If $\varsigma = 1$, then . . .
3. If $\varsigma < 1$, then . . ., where

$$U = \mathcal{K}(V_\mathrm{p}, W_\mathrm{m}). \tag{B.2}$$

References

[1] M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, Dover, 1965.

[2] R. Becker, Stoßbwelle und detonation (in German), Z. Physik 8 (1922) 321–362. [English transl.: Impact waves and detonation, Part I, N.A.C.A. Technical Memo. No. 505 (N.A.C.A., Washington, DC, 1929); Part II, N.A.C.A. Technical Memo. No. 506 (N.A.C.A., Washington, DC, 1929)].

[3] R.T. Beyer, The parameter B/A, in: M.F. Hamilton, D.T. Blackstock (Eds.), Nonlinear Acoustics, Academic Press, 1998, pp. 25–39.

[4] D.T. Blackstock, Approximate equations governing finite-amplitude sound in thermoviscous fluids (Technical Report), GD/E Report GD-1463-52, General Dynamics Corp., Rochester, NY, 1963, Chap. IV.

[5] R.L. Burden, J.D. Faires, Numerical Analysis, 5th edn., PSW–Kent, 1993, § 12.3.

[6] S. Chapman, T.G. Cowling, The Mathematical Theory of Non-Uniform Gases, 3rd edn., Cambridge University Press, 1970.

[7] S. Chen, et al., Camassa–Holm equations as a closure model for turbulent channel and pipe flow, Phys. Rev. Lett. 81 (1998) 5338–5341.

[8] C.I. Christov, On frame indifferent formulation of the Maxwell–Cattaneo model of finite-speed heat conduction, Mech. Res. Commun. 36 (2009) 481–486.

[9] M.S. Cramer, Numerical estimates for the bulk viscosity of ideal gases, Phys. Fluids 24 (2012) 066102.

[10] M. Destrade, G. Saccomandi, Nonlinear transverse waves in deformed dispersive solids. Wave Motion 45 (2008) 325–336.

[11] B.O. Enflo, C.M. Hedberg, O.V. Rudenko, Standing and propagating waves in cubically nonlinear media, in: B. Nilsson, L. Fishman (Eds.), AIP Conference Proceedings, vol. 834, AIP, 2006, pp. 187–195.

[12] P.M. Jordan, A note on the Lambert W-function: Applications in the mathematical and physical sciences. In: A.B. Gumel (Ed.), Mathematics of Continuous and Discrete Dynamical Systems, Contemporary Mathematics, Vol. 618, American Mathematical Society, 2014, pp. 247–263.

[13] P.M. Jordan, A survey of weakly-nonlinear acoustic models: 1910–2009, Mech. Res. Commun. 73 (2016) 127–139.

[14] H. Lamb, Hydrodynamics, 6th edn., Dover, 1945.

[15] M. Morduchow, P.A. Libby, On a complete solution of the one-dimensional flow equations of a viscous, heat-conducting, compressible gas, J. Aeronaut. Sci. 16 (1949) 674–684 and 704.

[16] D.A. Nield, A. Bejan, Convection in Porous Media, 2nd edn., Springer, 1999, Chap. 1.

[17] J.R. Ockendon, et al., Applied Partial Differential Equations, revised edn., Oxford University Press, 2003, Chap. 8.

[18] A.R. Rasmussen, Thermoviscous Model Equations in Nonlinear Acoustics: Analytical and Numerical Studies of Shocks and Rarefaction Waves, PhD Thesis, Dept. of Mathematics, Technical University Denmark, 2009.

[19] J.M. Reese, et al., The inner shock structure determined from a modified frame-independent second-order kinetic theory, in: R. Brun, L.Z. Dumitrescu (Eds.), Shock Waves @ Marseille IV: Shock Structure and Kinematics, Blast Waves and Detonations, Springer, 1995, pp. 51–56.

[20] M. Roy, Sur la structure de l'onde de choc, limite d'une quasi-onde de choc dans un fluide compressible et visqueux (in French), C. R. Acad. Sci. 218 (1944) 813–816.

[21] B. Straughan, Stability and Wave Motion in Porous Media, in: Applied Mathematical Sciences, vol. 165, Springer, 2008.

[22] G.I. Taylor, The conditions necessary for discontinuous motion in gases, Proc. R. Soc. Lond. A 84 (1910) 371–377.

[23] C. Truesdell, R.A. Toupin, The classical filed theories, in: S. Flügge (Ed.), Handbuch der Physik, vol. III/1, Springer, 1960, pp. 491–529.

[24] R. von Mises, Mathematical Theory of Compressible Fluid Flow, in: F.N. Frenkiel (Ed.), Applied Mathematics and Mechanics Series, vol. 3, Academic Press, 1958.

[25] G.B. Whitham, Linear and Nonlinear Waves, Wiley, 1974, § 6.3.